WENDROFF TYPE INEQUALITIES ON TIME SCALES VIA PICARD OPERATORS

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Abstract. Recently (see [11]) R.A.C. Ferreira, D.F.M. Torres proved some linear and nonlinear Wendroff type inequalities on time scales. Similar results were proved also by D.R. Anderson ([2] and [3]). It is well known (see [9]) that the Wendroff inequality is not the best possible upper estimate for the solutions of the integral inequality. The aim of our paper is to improve the known Wendroff type inequalities on time scales and to give a different proof for the existing inequalities. This improvement is motivated also by the work of A. Abdeldaim and M. Yakout (see [1] and [5]). The method we use is based on a variant of the abstract comparison Gronwall lemma (see [18], [15]) and on the theory of Picard operators ([16]).

1. Introduction

1.1. **Time scale analysis.** The time scale calculus was founded by Stefan Hilger in his PhD thesis (see [12]) as a unification of the classical real analysis, the q-calculus and the theory of difference equations. Since then this theory has been extensively studied in order to obtain a better understanding and a unified viewpoint of mathematical phenomenons occurring in the theory of difference equations and in the theory of differential equations. For an excellent introduction to the calculus on time scales and to the theory of dynamic equations on time scales we recommend the books [7] and [8] by M. Bohner and A. Peterson. Throughout the paper we use the basic notations from these books.

1.1.1. *Wendroff type inequalities on time scales.* In what follows we assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales with at least two points and we consider the time scale intervals $\tilde{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\tilde{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$ and $a_2 \in \mathbb{T}_2$. Let us denote $D = \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$. We also use the notation $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $e_p(t, s)$ denotes the usual exponential function on time scales with $p \in \mathcal{R}$, where p is a regressive function (see [7]). In [11] the authors obtained the following results:

Theorem 1.1. *(Theorem 2.1. in* [11]*) Let $u(t_1, t_2)$, $w(t_1, t_2)$, $a(t_1, t_2) \in C(D, \mathbb{R}_0^+)$ with $w(t_1, t_2)$ nondecreasing in each of its variables. If*

$$u(t_1, t_2) \le w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{1.1}$$

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for $(t_1, t_2) \in D$, *then*

$$u(t_1,t_2) \le w(t_1,t_2) e_{\int_{a_2}^{t_2} a(t_1,s_2)\Delta_2 s_2}(t_1,a_1), \quad (t_1,t_2) \in D. \tag{1.2}$$

Theorem 1.2. *(Theorem 2.2. in* [11]*) Let* $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$, *with* $w(t_1,t_2)$ *and* $a(t_1,t_2)$ *nondecreasing in each of the variables and* $g(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$, *where* $S = \{(t_1,t_2,s_1,s_2) \in D \times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ *and* g *is nondecreasing in the first two variables. If* u *satisfies the condition*

$$u(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2) u(s_1,s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{1.3}$$

for $(t_1, t_2) \in D$, *then*

$$u(t_1,t_2) \le w(t_1,t_2) e_{\int_{a_2}^{t_2} a(t_1,t_2) g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1), \ \forall (t_1,t_2) \in D. \tag{1.4}$$

In what follows we improve (1.2) and (1.4). The improved versions imply also improved bounds in the nonlinear cases (see theorem 3.1 and 3.2 in [11]). Applying the same technique we can obtain new (and simple) proofs for the previous theorems too.

1.2. **Picard operators.** The Picard operator technique was applied by many authors to study some functional nonlinear integral equations, see for example [4], [15], [16], [17]. We use the terminologies and notations from [15], [16], [17].

Let $(X, \to)$ be an L-space ([16]), $A : X \to X$ an operator. We denote by F_A the fixed points of A. We also denote $A^0 := 1_X$, $A^1 := A, \ldots, A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$ the iterate operators of the operator A.

Definition 1.3 ([15], [16], [17])**.** A is a Picard operator (briefly PO), if there exists $x_A^* \in X$ such that:
(i) $F_A = \{x_A^*\}$;
(ii) $A^n(x) \to x_A^*$ as $n \to \infty, \forall x \in X$.

In the following we recall two abstract Gronwall lemmas.

Lemma 1.4. *(*[15], [16]*)(Abstract Gronwall lemma) Let* $(X, \to, \le)$ *be an ordered L-space and* $A : X \to X$ *an operator. We assume that:*
(i) A *is PO;*
(ii) A *is increasing.*
If we denote by x_A^* *the unique fixed point of* A*, then:*
(a) $x \le A(x) \Rightarrow x \le x_A^*$;
(b) $x \ge A(x) \Rightarrow x \ge x_A^*$.

Lemma 1.5. *(*[15], [16]*)(Abstract Gronwall-comparison lemma) Let* $(X, \to, \le)$ *be an ordered L-space and* $A_1, A_2 : X \to X$ *be two operators. We assume that:*
(i) A_1 *is increasing;*
(ii) A_1 *and* A_2 *are POs;*
(iii) $A_1 \le A_2$.
If we denote by x_2^* *the unique fixed point of* A_2*, then*

$$x \le A_1(x) \Rightarrow x \le x_2^*.$$

These lemmas are very powerful because once we prove that the operator is Picard operator and we have an L-space structure, the Gronwall type inequalities can be proved without any additional effort (calculation). In many Gronwall type inequalities the upper bound of the solution is the solution of the corresponding fixed point equation. These can be proved using lemma 1.4. This is not the case of the Wendroff type inequalities, where the upper bound is not the solution of the corresponding fixed point equation (see [9]). To handle these cases lemma 1.5 can be used (see [18]). The main difficulty in using lemma 1.5 is the construction of the operator A_2. To avoid this we propose the following variant:

Lemma 1.6. *(Abstract Gronwall lemma) Let* $(X, \to, \leq)$ *be an orderd L-space and* $A : X \to X$ *be an operator with the following properties:*
(i) A *is increasing;*
(ii) A *is PO;*
(iii) there exists $\overline{x}$ *with the property* $A\overline{x} \leq \overline{x}$.
If for some $x \in X$ *we have* $x \leq Ax$, *then* $x \leq \overline{x}$.

Proof. A is increasing, so the inequality $x \leq Ax$ implies $x \leq A^n x$, $\forall n \in \mathbb{N}$. Due to the Picard property of the operator A this implies $x \leq x^*$, where x^* is the unique solution of the operator A. On the other hand the inequality $A\overline{x} \leq \overline{x}$ implies $A^n\overline{x} \leq \overline{x}$ and so $x^* \leq \overline{x}$, which completes the proof. □

Remark 1.7. If the conditions of lemma 1.4 or 1.5 are satisfied, than the conditions of lemma 1.6 are also satisfied. From this viewpoint lemma 1.6 is more general than lemma 1.4 and lemma 1.5. We have to mention that in many cases the inequality $A\overline{x} \leq \overline{x}$ can be established by using the operator A_2 with the properties $A \leq A_2$ and $A_2\overline{x} = \overline{x}$. Our result from theorem 3.2 can not be proved with this technique because the operator A_2 for which $A_2\overline{x} = \overline{x}$ does not satisfy $A \leq A_2$. This motivates the necessity of lemma 1.6.

2. Preliminary results

In this section we extend the metric introduced by C.C. Tisdell and A. Zaidi in [19] to functions with several variables. This allows us to prove that our operators are Picard operators, in fact they are contractions if we use a well chosen metric. Suppose that $\alpha, \beta > 0$ are real constants and define the functionals

(2.1)
$$d_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \times C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)) \to \mathbb{R}$$
by

$$d_{\alpha,\beta} = \sup_{\substack{s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \\ s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}}} \frac{\|u(s_1, s_2) - v(s_1, s_2)\|}{e_\alpha(s_1, a_1) \cdot e_\beta(s_2, a_2)} \tag{2.2}$$

for all $u, v \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ and

$$\| \cdot \|_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \to \mathbb{R} \tag{2.3}$$

$$\|u\|_{\alpha,\beta} = \sup_{\substack{s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \\ s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}}} \frac{\|u(s_1, s_2)\|}{e_\alpha(s_1, a_1) \cdot e_\beta(s_2, a_2)} \tag{2.4}$$

for all $u \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$, where $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$.

Lemma 2.1. *If* $\alpha, \beta > 0$, *and* $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, *we have the following properties:*

(1) $d_{\alpha,\beta}$ *is a metric on* $C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2},\mathbb{R}^n)$;
(2) $C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2},\mathbb{R}^n)$ *is a complete metric space with* $d_{\alpha,\beta}$;
(3) $\|\cdot\|_{\alpha,\beta}$ *is a norm on* $C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2},\mathbb{R}^n)$ *and it is equivalent to* $\|\cdot\|_{0,0}$.
(4) $C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2},\mathbb{R}^n,\|\cdot\|_{\alpha,\beta})$ *is a Banach space.*

The proof of this lemma is quite straightforward, so we omit it. For the simplicity of notations in what follows we denote $C([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2},\mathbb{R})$ by X. Using this Bielecki type (or "TZ") metric, we prove the following properties:

Theorem 2.2. *If* $w,a\in X$, $\sigma_1(b_1)<\infty,\sigma_2(b_2)<\infty$, *the operator* $A_1:X\to X$ *defined by*

$$A_1(u)(t_1,t_2)=w(t_1,t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2}a(s_1,s_2)u(s_1,s_2)\Delta_1s_1\Delta_2s_2, \tag{2.5}$$

is well defined and there exist $\alpha,\beta>0$ *such that* A_1 *is a contraction on* $(X,d_{\alpha,\beta})$.

Theorem 2.3. *If* $w,a\in X$, g *is continuous,* $\sigma_1(b_1)<\infty,\sigma_2(b_2)<\infty$, *the operator* $A_2:X\to X$ *defined by*

$$A_2(u)(t_1,t_2)=w(t_1,t_2)+a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2}g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1s_1\Delta_2s_2, \tag{2.6}$$

is well defined and there exist $\alpha,\beta>0$ *such that* A_2 *is a contraction on* $(X,d_{\alpha,\beta})$.

Proof of theorem 2.2. Denote by M the maximum of $a(s_1,s_2)$ if $s_1\in[a_1,\sigma_1(b_1)]_{\mathbb{T}_1}$ and $s_2\in[a_2,\sigma_2(b_2)]_{\mathbb{T}_2}$. Due to the given conditions M exists and $M<\infty$.

$$\begin{aligned}|A_1(u)(t_1,t_2)-A_1(v)(t_1,t_2)|\le&\int_{a_1}^{t_1}\int_{a_2}^{t_2}|a(s_1,s_2)||u(s_1,s_2)-v(s_1,s_2)|\Delta_1s_1\Delta_2s_2\\ \le& M\int_{a_1}^{t_1}\int_{a_2}^{t_2}\frac{|u(s_1,s_2)-v(s_1,s_2)|}{e_\alpha(s_1,a_1)e_\beta(s_2,a_2)}e_\alpha(s_1,a_1)e_\beta(s_2,a_2)\Delta_1s_1\Delta_2s_2\\ \le& M\|u-v\|_{\alpha,\beta}\int_{a_1}^{t_1}\int_{a_2}^{t_2}e_\alpha(s_1,a_1)e_\beta(s_2,a_2)\Delta_1s_1\Delta_2s_2\\ \le&\frac{M}{\alpha\beta}\|u-v\|_{\alpha,\beta}e_\alpha(t_1,a_1)e_\beta(t_2,a_2).\end{aligned}$$

The last inequality implies

$$\|A_1(u)-A_1(v)\|_{\alpha,\beta}\le\frac{M}{\alpha\beta}\|u-v\|_{\alpha,\beta}, \tag{2.7}$$

so A_1 is a contraction on X if $\alpha\beta>M$. □

Remark 2.4. The proof of theorem 2.3 can be done in a similar way by using the maximum of g on $([a_1,\sigma_1(b_1)]_{\mathbb{T}_1}\times[a_2,\sigma_2(b_2)]_{\mathbb{T}_2})^2$.

Remark 2.5. We can obtain the contractive property of a more general nonlinear operator $A_3:X\to X$ defined by

$$A_3(u)(t_1,t_2)=w(t_1,t_2)+a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2}f(t_1,t_2,s_1,s_2,u(s_1,s_2))\Delta_1s_1\Delta_2s_2,$$

where f is continuous and has the Lipschitz property in the last variable.

Remark 2.6. Due to theorem 2.2 and 2.3 the operators A_1 and A_2 are Picard operators.

In the calculations we need the following two properties:

Lemma 2.7. *If f is continuous and is continuously Δ differentiable with respect to t, then the function*

$$U(t) = \int_a^t f(s,t)\Delta s$$

admits a Δ derivative with respect to t and

$$U^{\Delta}(t) = \int_a^t \frac{\partial f}{\Delta t}(s,t)\Delta s + f(t,t).$$

Lemma 2.8. *If $f : E \to \mathbb{R}$ is a continuous function, where*

$$E = \{(s,t) \in \mathbb{T}_1 \times \mathbb{T}_2 | a \leq t < b, a \leq s < t\},$$

then the function $g : [a,b) \to \mathbb{R}$, defined by

$$g(t) = \int_a^t f(s,t)\Delta_1 s$$

is Δ integrable on $[a,b)$ and we have

$$\int_a^b \int_a^t f(s,t)\Delta_1 s \Delta_2 t = \int_a^b \int_{\sigma(s)}^b f(s,t)\Delta_2 t \Delta_1 s.$$

Remark 2.9. If $f : E \to \mathbb{R}$ is a continuous function, where

$$E = \{(t_1,t_2,s_1,s_2) \in (\mathbb{T}_1 \times \mathbb{T}_2)^2 | a_1 \leq s_1 < t_1, a_2 \leq s_2 < t_2\},$$

then the function $g : [a_1,t_1) \times [a_2,t_2) \to \mathbb{R}$, defined by

$$g(s_1,s_2) = \int_{a_1}^{s_1} \int_{a_2}^{s_2} f(t_1,t_2,\xi_1,\xi_2)\Delta_1\xi_1\Delta_2\xi_2$$

is Δ integrable on $[a_1,t_1) \times [a_2,t_2)$ and we have

$$\int_{a_1}^{t_1} \int_{a_2}^{t_2} g(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 =$$

$$= \int_{a_1}^{t_1} \int_{a_2}^{t_2} \int_{\sigma_1(s_1)}^{t_1} \int_{\sigma_2(s_2)}^{t_2} f(\xi_1,\xi_2,s_1,s_2)\Delta_1\xi_1\Delta_2\xi_2\Delta_1 s_1\Delta_2 s_2.$$

3. Main results

3.1. **Linear inequalities.** In this section we give new estimates for u and we prove that these are better than (1.2), (1.4). We need the following lemma

Lemma 3.1. *For the function $V : E \to \mathbb{R}$, defined by*

$$V(t_1,t_2,s_1,s_2) = e_{\int_{s_2}^{t_2} a(t_1,\xi_2)\Delta_2\xi_2}(t_1,s_1),$$

where

$$E = \{(t_1,t_2,s_1,s_2) \in (\mathbb{T}_1 \times \mathbb{T}_2)^2 | a_1 \leq s_1 < t_1, a_2 \leq s_2 < t_2\}$$

we have

$$a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)) \leq \frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2). \tag{3.1}$$

and

$$a(t_1,t_2)V(t_1,t_2,s_1,s_2) \le \frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1,t_2,s_1,s_2). \tag{3.2}$$

Proof. The function V is Δ_1 differentiable with respect to s_1 and we have

$$\frac{\partial V}{\Delta_1 s_1}(t_1,t_2,s_1,s_2) = -\int_{s_2}^{t_2} a(s_1,\xi_2)\Delta_2\xi_2 \cdot V(t_1,t_2,\sigma_1(s_1),s_2).$$

Moreover the function $\frac{\partial V}{\Delta_1 s_1}$ is Δ_2 differentiable and we have

$$\frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2) = \int_{s_2}^{t_2} a(s_1,\xi_2)\Delta_2\xi_2 \int_{\sigma_1(s_1)}^{t_1} a(\xi_1,s_2)\Delta_1\xi_1 \cdot V(t_1,t_2,\sigma_1(s_1),\sigma(s_2)) +$$
$$+ a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)).$$

Since the function a is nonnegative we obtain

$$a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)) \le \frac{\partial^2 V}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2).$$

Using a similar argument we have

$$\frac{\partial V}{\Delta_1 t_1}(t_1,t_2,s_1,s_2) = \int_{s_2}^{t_2} a(t_1,\xi_2)\Delta_2\xi_2 \cdot V(t_1,t_2,s_1,s_2).$$

The function $\frac{\partial V}{\Delta_1 t_1}$ is Δ_2 differentiable with respect to t_2 and we have

$$\frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1,t_2,s_1,s_2) = \int_{s_2}^{\sigma_2(t_2)} a(t_1,\xi_2)\Delta_2\xi_2 \int_{s_1}^{t_1} a(\xi_1,t_2)\Delta_1\xi_1 \cdot V(t_1,t_2,s_1,s_2) +$$
$$+ a(t_1,t_2)V(t_1,t_2,s_1,s_2).$$

Since the function a is nonnegative we obtain

$$a(t_1,t_2)V(t_1,t_2,s_1,s_2) \le \frac{\partial^2 V}{\Delta_1 t_1 \Delta_2 t_2}(t_1,t_2,s_1,s_2).$$

□

Theorem 3.2. *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If $u(t_1,t_2)$ satisfies*

$$u(t_1,t_2) \le w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{3.3}$$

for $(t_1,t_2) \in D$, then

(3.4)

$$u(t_1,t_2) \le w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1 \Delta_2 s_2,$$

for $(t_1,t_2) \in D$, where σ_1 and σ_2 are the jump operators on $\mathbb{T}_1$ respectively $\mathbb{T}_2$.

Proof. The integral operator $A : C(D) \to C(D)$ defined by

$$A(u)(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2, \tag{3.5}$$

is a Picard operator (due to theorem 2.2). Moreover the space $(C(D), \|\cdot\|)$ is an ordered Banach space with the natural ordering

$$u \le v \Leftrightarrow u(t_1,t_2) \le v(t_1,t_2),\ \forall (t_1,t_2) \in D$$

and the operator A is an increasing operator, so the inequality $u \le Au$ implies $u \le u^*$, where u^* is the unique solution of the equation $Au = u$. On the other hand it is easy to check that the unique fixed point of A is not the function

$$\overline{u}(t_1,t_2) = w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2,$$

so by lemma 1.6 we need to prove $A\overline{u} \le \overline{u}$. Using the function V from lemma 3.1 it is sufficient to prove

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 + \tag{3.6}$$

$$+\int_{a_1}^{t_1}\int_{a_2}^{t_2}\int_{a_1}^{s_1}\int_{a_2}^{s_2} a(s_1,s_2)a(\xi_1,\xi_2)w(\xi_1,\xi_2)V(s_1,s_2,\sigma_1(\xi_1),\sigma_2(\xi_2))\Delta_1\xi_1\Delta_2\xi_2\Delta_1 s_1\Delta_2 s_2 \le \tag{3.7}$$

$$\le \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2. \tag{3.8}$$

Changing the order of integration in (3.7) and renaming the variables it is sufficient to prove

$$1 + \int_{\sigma_1(s_1)}^{t_1}\int_{\sigma_2(s_2)}^{t_2} a(\xi_1,\xi_2)V(\xi_1,\xi_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1\xi_1\Delta_2\xi_2 \le V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)).$$

This can be obtained integrating (3.2) from $\sigma_1(s_1)$ to t_1 and than from $\sigma_2(s_2)$ to t_2. □

Theorem 3.3. *If the conditions of Theorem 3.2 are satisfied, the estimation of the Theorem 3.2 is better than the estimation from Theorem 1.1.*

Proof. Integrating inequality (3.1) with respect to s_1 and s_2 on the rectangle $[a_1,t_1)_{\mathbb{T}_1} \times [a_2,t_2)_{\mathbb{T}_2}$ we deduce

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le$$

$$\le \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{\partial^2 V}{\Delta_1 s_1\Delta_2 s_2}(t_1,t_2,s_1,s_2)\Delta_1 s_1\Delta_2 s_2 =$$

$$= V(t_1,t_2,t_1,t_2) - V(t_1,t_2,a_1,t_2) - V(t_1,t_2,t_1,a_2) + V(t_1,t_2,a_1,a_2).$$

But $V(t_1,t_2,t_1,t_2) = V(t_1,t_2,a_1,t_2) = V(t_1,t_2,t_1,a_2) = 1$, so we obtain

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)V(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le V(t_1,t_2,a_1,a_2) - 1. \tag{3.9}$$

The function w is nonnegative and nondecreasing in both variables, hence we have

$$w(t_1,t_2) + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)w(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2 \le$$

$$w(t_1,t_2)\left(1 + \int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} a(s_1,s_2)e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2\right) \le$$

$$\le w(t_1,t_2)V(t_1,t_2,a_1,a_2).$$

This inequality shows that the estimation in Theorem 3.2 is better than the estimation from Theorem 1.1. □

Lemma 3.4. *For the function $W : E \to \mathbb{R}$, defined by*

$$W(t_1,t_2,s_1,s_2) = e_{\int_{s_2}^{t_2} a(t_1,t_2)g(t_1,t_2,t_1,\eta)\Delta_2\eta}(t_1,s_1),$$

where

$$E = \{(t_1,t_2,s_1,s_2) \in (\mathbb{T}_1\times\mathbb{T}_2)^2 | a_1 \le s_1 < t_1, a_2 \le s_2 < t_2\}$$

we have

$$a(t_1,t_2)g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2)) \le \frac{\partial^2 W}{\Delta_1 s_1\Delta_2 s_2}(t_1,t_2,s_1,s_2) \tag{3.10}$$

Remark 3.5. The proof of the Lemma 3.4 is similar to the proof of the Lemma 3.1.

Theorem 3.6. *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D\times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ and g is nondecreasing in the first two variables. If u satisfies the condition*

$$u(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2, \tag{3.11}$$

for $(t_1,t_2) \in D$, then

$$\begin{aligned} u(t_1,t_2) \le & w(t_1,t_2) + \\ & a(t_1,t_2)\int\limits_{a_1}^{t_1}\int\limits_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)w(s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2, \end{aligned} \tag{3.12}$$

for $(t_1,t_2) \in D$, where σ_1 and σ_2 are the jump operators on $\mathbb{T}_1$ respectively $\mathbb{T}_2$.

Proof. We apply the same technique as in [11]. We consider that t_1^* and t_2^* are fixed and we consider the operator $A^* : C(D) \to C(D)$ defined by

$$A^*(u)(t_1,t_2) = w(t_1,t_2) + a(t_1^*,t_2^*)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1^*,t_2^*,s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2. \tag{3.13}$$

It is clear that if u satisfies the conditions of theorem 3.6, then $u(t_1,t_2) \le A^*(u)(t_1,t_2)$, for $t \le t_1^*$ and $t_2 \le t_2^*$. t_1^*, t_2^* beeing fixed, theorem 3.2 implies

$$u(t_1,t_2) \le w(t_1,t_2)+ \tag{3.14}$$

$$+\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(t_1^*,t_2^*)g(t_1^*,t_2^*,s_1,s_2)w(s_1,s_2)H(s_1,s_2,t_1,t_2,t_1^*,t_2^*)\Delta_1 s_1\Delta_2 s_2,$$

where

$$H(s_1,s_2,t_1,t_2,t_1^*,t_2^*) = e_{\int_{\sigma_2(s_2)}^{t_2} a(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))$$

This inequality is valid for $t_1 = t_1^*$ and $t_2 = t_2^*$ and t_1^*, t_2^* are arbitrary, so we obtain (3.12). □

Theorem 3.7. *If the conditions of Theorem 3.6 are satisfied, the estimation of the Theorem 3.6 is better than the estimation from Theorem 1.2.*

Proof. Integrating inequality (3.10) from a_1 to t_1 and from a_2 to t_2 with respect to s_1 and s_2 on the rectangle $[a_1,t_1)_{\mathbb{T}_1} \times [a_2,t_2)_{\mathbb{T}_2}$ we have

$$\int_{a_1}^{t_1}\int_{a_2}^{t_2} a(t_1,t_2)g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le$$

$$\le \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{\partial^2 W}{\Delta_1 s_1 \Delta_2 s_2}(t_1,t_2,s_1,s_2)\Delta_1 s_1\Delta_2 s_2 =$$

$$= W(t_1,t_2,t_1,t_2) - W(t_1,t_2,a_1,t_2) - W(t_1,t_2,t_1,a_2) + W(t_1,t_2,a_1,a_2).$$

But $W(t_1,t_2,t_1,t_2) = W(t_1,t_2,a_1,t_2) = W(t_1,t_2,t_1,a_2) = 1$, so we obtain

$$a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le \tag{3.15}$$

$$\le W(t_1,t_2,a_1,a_2) - 1.$$

The function w is nonnegative and nondecreasing in both variables, hence we have

$$w(t_1,t_2)+a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)w(s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2 \le$$

$$w(t_1,t_2)\left(1 + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)W(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1 s_1\Delta_2 s_2\right) \le$$

$$\le w(t_1,t_2)W(t_1,t_2,a_1,a_2).$$

This inequality shows that the estimation in Theorem 3.6 is better than the estimation from Theorem 1.2. □

3.2. **Nonlinear inequalities.** In this subsection we give improved estimations to the recently proved nonlinear integral inequalities in ([11]) combining the method from ([11]) with theorem 3.2 and 3.6. First we recall the nonlinear integral inequalities from [11]:

Theorem 3.8. *(Theorem 3.1 in* [11]*) Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 \tag{3.16}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq w^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} a(t_1,s_2)w^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}}, (t_1,t_2) \in D. \tag{3.17}$$

Theorem 3.9. *(Theorem 3.2 in* [11]*) Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S,\mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D \times D : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$ and g is nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and is u satisfies the condition*

$$u^p(t_1,t_2) \leq w(t_1,t_2) + a(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2, \tag{3.18}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq w^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} a(t_1,t_2)w^{\frac{q}{p}-1}(t_1,s_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}}, \ \forall (t_1,t_2) \in D. \tag{3.19}$$

In what follows we prove the following improvements of these to theorems:

Theorem 3.10. *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D,\mathbb{R}_0^+)$ with $w(t_1,t_2)$ nondecreasing in each of its variables. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq w(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} a(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 \tag{3.20}$$

for $(t_1,t_2) \in D$, then

$$u(t_1,t_2) \leq \left[w(t_1,t_2) + w(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} H(t_1,t_2,s_1,s_2)\Delta_1 s_1 \Delta_2 s_2\right]^{\frac{1}{p}}, \tag{3.21}$$

where

$$H(t_1,t_2,s_1,s_2) = a(s_1,s_2)w^{\frac{q}{p}-1}(s_1,s_2)e_{\int_{\sigma(s_2)}^{t_2} a(t_1,\eta)w^{\frac{q}{p}-1}(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s1)),$$

$(t_1,t_2) \in D$.

Proof. Suppose $w(t_1,t_2) > 0$, $(t_1,t_2) \in D$. We denote u^p by $\overline{u}$. If u satisfies the conditions of the previous theorem, due to the monotonicity of w we obtain

$$\frac{\overline{u}(t_1,t_2)}{w(t_1,t_2)} \leq 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} \frac{a(s_1,s_2)}{w(s_1,s_2)}\overline{u}^{\frac{q}{p}}(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2,$$

hence for the function v defined by the right hand side of the previous inequality we have

$$\frac{\partial^2 v}{\Delta_1 t_1 \Delta_2 t_2} = \frac{a(t_1,t_2)}{w(t_1,t_2)} \overline{u}^{\frac{q}{p}}(t_1,t_2) \le a(t_1,t_2) w^{\frac{q}{p}-1}(t_1,t_2) v(t_1,t_2).$$

Integrating both sides we deduce that the function v satisfies the following inequality:

$$v(t_1,t_2) \le 1 + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1,s_2) w^{\frac{q}{p}-1}(s_1,s_2) v(s_1,s_2) \Delta_1 s_1 \Delta_2 s_2.$$

Aplying theorem 3.2 for v and using $u(t_1,t_2) \le (w(t_1,t_2)v(t_1,t_2))^{\frac{1}{p}}$ we obtain (3.21). □

Remark 3.11. If $w \ge 0$, we can replace w with $w_\varepsilon = w + \varepsilon$ and then consider $\varepsilon \to 0$.

Remark 3.12. Due to Theorem 3.3 the estimation in Theorem 3.10 is better than the estimation in Theorem 3.8.

Using the same argument as in the previous theorem we obtain the following result:

Theorem 3.13. *Let $u(t_1,t_2)$, $w(t_1,t_2)$, $a(t_1,t_2) \in C(D, \mathbb{R}_0^+)$, with $w(t_1,t_2)$ and $a(t_1,t_2)$ nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S, \mathbb{R}_0^+)$, where $S = \{(t_1,t_2,s_1,s_2) \in D \times D : a_1 \le s_1 \le t_1, a_2 \le s_2 \le t_2\}$ and g is nondecreasing in the first two variables. If p and q are two positive real numbers such that $p \ge q$ and u satisfies the condition*

$$u^p(t_1,t_2) \le w(t_1,t_2) + a(t_1,t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2) u^q(s_1,s_2) \Delta_1 s_1 \Delta_2 s_2, \tag{3.22}$$

for $(t_1,t_2) \in D$, then

(3.23)

$$u(t_1,t_2) \le w^{\frac{1}{p}}(t_1,t_2) \left[1 + a(t_1,t_2) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2) H(t_1,t_2,s_1,s_2) \Delta_1 s_1 \Delta_2 s_2 \right]^{\frac{1}{p}},$$

where

$$H(t_1,t_2,s_1,s_2) = w^{\frac{q}{p}-1}(s_1,s_2) e_{\int_{\sigma(s_2)}^{t_2} a(t_1,\eta) g(t_1,t_2,t_1,\eta) w^{\frac{q}{p}-1}(t_1,\eta) \Delta_2 \eta}(t_1, \sigma_1(s1)),$$

$(t_1,t_2) \in D$.

Remark 3.14. Due to Theorem 3.7 the estimation of the Theorem 3.13 is better than the estimation from Theorem 3.9.

Remark 3.15. Theorem 3.10 and Theorem 3.13 generalize and extend to time scales Theorem 2.1, Theorem 2.2 and Theorem 2.3. from [10].

3.3. **Applications.** What it follows we present some applications of our results form the Theorem 3.2.

Theorem 3.16. *Let us consider the the following partial delta dynamic equation*

$$\frac{\partial^2 u(t_1,t_2)}{\Delta_2 t_2 \Delta_1 t_1} = F(t_1,t_2,u(t_1,t_2)) \tag{3.24}$$

on the our domain D, equipped with the initial conditions

$$u(t_1,a_2)=g_1(t_1),\ u(a_1,t_2)=g_2(t_2),\forall t_1\in\tilde{\mathbb{T}}_1,\ t_2\in\tilde{\mathbb{T}}_2 \tag{3.25}$$

and we assume, that $F\in C(D\times\mathbb{R}_0^+,\mathbb{R}_0^+)$, $g_1\in C(\tilde{\mathbb{T}}_1,\mathbb{R}_0^+)$, $g_2\in C(\tilde{\mathbb{T}}_2,\mathbb{R}_0^+)$ *and* g_1 *and* g_2 *are nondecreasing.*

Now if we assume, that F *on its domain satisfies the*

$$F(t_1,t_2,u)\le f(t_1,t_2)u,\ \forall(t_1,t_2)\in D,\ u\in C(D,\mathbb{R}_0^+) \tag{3.26}$$

for the given function $f\in C(D,\mathbb{R}_0^+)$ *with the nondecreasing property on both of its variables.*

If u *is the solution of the initial value problem 3.24-3.25, then* u *satisfies the*

$$\begin{aligned}u(t_1,t_2)\le & g_1(t_1)+g_2(t_2)+\\ &+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\left(g_1(s_1)+g_2(s_2)\right)e_{\int_{\sigma_2(s_2)}^{t_2} f(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2\end{aligned}$$

inequality.

Proof. Let $u(t_1,t_2)$ be the solution of the initial value problem 3.24-3.25. Then it satisfies the equation

$$u(t_1,t_2)=g_1(t_1)+g_2(t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2}F(s_1,s_2,u(s_1,s_2))\Delta_1 s_1\Delta_2 s_2.$$

From 3.26 we have

$$u(t_1,t_2)\le g_1(t_1)+g_2(t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2}f(s_1,s_2)u(s_1,s_2)\Delta_1 s_1\Delta_2 s_2.$$

This inequality satisfies the requirements of the Theorem 3.2, so we have the following estimation with $w(t_1,t_2):=g_1(t_1)+g_2(t_2)$ and $a(t_1,t_2):=f(t_1,t_2)$ $\forall(t_1,t_2)\in D:$

$$\begin{aligned}u(t_1,t_2)\le & g_1(t_1)+g_2(t_2)+\\ &+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\left(g_1(s_1)+g_2(s_2)\right)e_{\int_{\sigma_2(s_2)}^{t_2} f(t_1,\eta)\Delta_2\eta}(t_1,\sigma_1(s_1))\Delta_1 s_1\Delta_2 s_2.\end{aligned}$$

So we have required inequality. □

Remark 3.17. If we have some other type of initial value problem instead of 3.24-3.25, for example

$$\frac{\partial^2 u^p(t_1,t_2)}{\Delta_2 t_2\Delta_1 t_1}=F(t_1,t_2,u(t_1,t_2)) \tag{3.27}$$

on the our domain D, equipped with the initial conditions

$$u^p(t_1,a_2)=g_1(t_1),\ u^p(a_1,t_2)=g_2(t_2),\forall t_1\in\tilde{\mathbb{T}}_1,\ t_2\in\tilde{\mathbb{T}}_2 \tag{3.28}$$

and we assume the same properties for F,g_1,g_2, and also

$$F(t_1,t_2,u)\le f(t_1,t_2)u^q,\ \forall(t_1,t_2)\in D,\ u\in C(D,\mathbb{R}_0^+), \tag{3.29}$$

for the arbitrary positive real numbers p and q with $p \geq q$, we could have similar estimate for the solution u of the initial value problem 3.27-3.28, using our result form Theorem 3.10.

Remark 3.18. Due to Theorem 3.3 we have, that our estimations for solution of the problem 3.24-3.25 respective 3.27-3.28, are better as the estimation from [11].

Now we give two concrete examples to the initial value problem 3.24-3.25.

Example 3.19. Let be our time scales $\mathbb{T}_1 := \mathbb{R}$ and $\mathbb{T}_2 = \mathbb{R}$, what means, that $D = [0, \infty) \times [0, \infty)$ and the problem 3.24-3.25 is considered as a real continuous initial value problem.

Now let be $F(t_1, t_2, u(t_1, t_2)) := \sin(t_1 t_2 u(t_1, t_2))u(t_1, t_2)$ on $D \times \mathbb{R}_0^+$ and the initial conditions: $u(t_1, 0) = g_1(t_1) := 0$, $\forall t_1 \in [0, \infty)$ and $u(0, t_2) = g_2(t_2) := 1$, $\forall t_2 \in [0, \infty)$.

It is clear, that $F(t_1, t_2, u) \leq u$ $(f(t_1, t_2) \equiv 1)$ on the whole of its domain, and all of the conditions from Theorem 3.16 are satisfied. So applying this theorem we have on the domain D:

$$\begin{aligned} u(t_1, t_2) \leq & 1 + \int_0^{t_1} \int_0^{t_2} \exp\left(\int_{s_1}^{t_1} \int_{s_2}^{t_2} 1 d\eta d\xi \right) ds_2 ds_1 \\ = & 1 + \int_0^{t_1} \int_0^{t_2} \exp\left((t_1 - s_1)(t_2 - s_2) \right) ds_2 ds_1 \end{aligned}$$

On the other hand, if we use the estimate from Theorem 1.1, we have

$$u(t_1, t_2) \leq \exp(t_1 t_2).$$

If we compare these two estimations, we can see, that the difference between them increases exponentially. As the following table shows, we have calculates numerically the values of

$$h_1(t_1, t_2) := 1 + \int_0^{t_1} \int_0^{t_2} \exp\left((t_1 - s_1)(t_2 - s_2) \right) ds_2 ds_1,$$

and

$$h_2(t_1, t_2) := \exp(t_1 t_2)$$

and also the difference $h_2(t_1, t_2) - h_1(t_1, t_2)$ on a grid with node points

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 4,$$

$$0 = y_0 < y_1 < \cdots < y_{n-1} < y_n = 4,$$

where $\Delta x = \Delta y = 0.5$.

The values of $h_2(t_1, t_2) - h_1(t_1, t_2)$ are

t_1/t_2	0	0.5	1	1.5	2	2.5	3	3.5	4
0	0	0	0	0	0	0	0	0	0
0.5	0	0.01749	0.07857	0.1992	0.40038	0.70965	1.1631	1.8078	2.7052
1	0	0.07857	0.40038	1.1631	2.7052	5.6022	10.828	20.02	35.931
1.5	0	0.1992	1.1631	3.9353	10.828	26.903	63.165	143.4	318.81
2	0	0.40038	2.7052	10.828	35.931	109.41	318.81	906.65	2542.2
2.5	0	0.70965	5.6022	26.903	109.41	414.74	1520.2	5476.4	19536
3	0	1.1631	10.828	63.165	318.81	1520.2	7067	32434	1.478e+005
3.5	0	1.8078	20.02	143.4	906.65	5476.4	32434	1.9021e+005	1.1094e+006
4	0	2.7052	35.931	318.81	2542.2	19536	1.478e+005	1.1094e+006	8.2906e+006

From these numerical results on this initial value problem it can see, that the estimations from Theorem 3.2 are much sharp as in the Theorem 1.1.

Example 3.20. Let be the time scales $\mathbb{T}_1 := \mathbb{Z}$ and $\mathbb{T}_2 = \mathbb{Z}$, what means, that $D = \mathbb{N}_0 \times \mathbb{N}_0$ and the problem 3.24-3.25 is considered as a discrete initial value problem:

$$u(m,n) = g_1(m) + g_2(n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u(s,t)).$$

Now let be $F(m,n,u(m,n)) := u(m,n)$ on $D \times \mathbb{R}_0^+$ and the initial conditions: $u(m,0) = g_1(m) := 0,\ \forall m \in \mathbb{N}_0$ and $u(0,n) = g_2(n) := 1,\ \forall n \in \mathbb{N}_0$.

It is clear, that $F(m,n,u) \le u$ $(f(m,n) \equiv 1)$ on the whole of its domain, and all of the conditions from Theorem 3.16 are satisfied. So applying this theorem we have on the domain D:

$$\begin{aligned} u(m,n) \le & 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left[\prod_{\eta=t}^{n-1}\left(1+\sum_{\xi=s}^{m-1}1\right)\right] = 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left[\prod_{\eta=t}^{n-1}(1+m-s)\right] \\ = & 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left[(1+m-s)^{n-t}\right] = 1 + \sum_{s=0}^{m-1}\left[(1+m-s)\frac{(1+m-s)^n-1}{m-s}\right] \end{aligned}$$

On the other hand dealing with the Theorem 1.1 we have the following estimation in this case:

$$u(m,n) \le \prod_{t=0}^{n-1}(1+m) = (1+m)^n$$

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